

Generic two-phase coexistence in nonequilibrium systems

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Abstract. A beautifully simple model introduced a couple of decades ago, Toom's cellular automaton, revealed that non-equilibrium systems may exhibit generic bistability, i.e. two-phase coexistence over a finite area of the (two-dimensional) phase diagram, in violation of the equilibrium Gibbs phase rule. In this paper we analyse two interfacial models, describing more realistic situations, that share with Toom's model a phase diagram with a broad region of phase coexistence. An analysis of the interfacial models yields conditions for generic bistability in terms of physically relevant parameters that may be controlled experimentally.

PACS. 05.70.Fh Phase transitions: general studies – 05.40.Ca Noise – 05.50.+q Lattice theory and statistics (Ising, Potts, etc.)

1 Introduction

Gibbs' phase rule states that two-phase coexistence of a single-component system, characterized by an n - dimensional parameter-space, may occur in an $n - 1$ - dimensional region. For example, the two equilibrium phases of the Ising model coexist on a line in the temperature-magnetic-field phase diagram. Nonequilibrium systems may violate this rule and several models, where phase coexistence occurs over a finite (n -dimensional) region of the parameter space, have been reported. The first example of this behaviour was found in Toom's model [1–3], that exhibits *generic bistability*, i.e. two-phase coexistence over a finite region of its two-dimensional parameter space (see Sect. 2). In addition to its interest as a genuine nonequilibrium property, generic multistability, defined as a generalization of bistability, is both of practical and theoretical relevance. In particular, it has been used recently to argue that some complex structures appearing in nature could be truly stable rather than metastable (with important applications in theoretical biology), and as the theoretical basis for an error-correction method in computer science (see [3,4] for an illuminating and pedagogical discussion of these ideas).

The necessary ingredients to generate broad (n - dimensional) phase-coexistence in Toom's model have been discussed in [2], where general criteria were also identified. The main idea is that, to obtain generic bistability,

a nonequilibrium mechanism that prevents the growth of the stable phase while enhancing the stability of the other one, must exist. This is achieved in Toom's model by introducing an explicit, somewhat artificial, asymmetry in the dynamic rules.

Recent progress in nonequilibrium statistical mechanics has led to the analysis of more realistic, physically motivated, models that also exhibit generic bistability. In this paper we describe two such examples, both interfacial models of pinning-depinning transitions. The analogies and differences with Toom's model are described in detail providing the reader with a general view of the relevant nonequilibrium physical mechanisms. In particular, we identify the ingredients, of the physically motivated models, that are responsible for the emergence of generic multistability.

This paper is organized as follows. In Section 2 we review Toom's model. The following two sections are devoted to the description of the two interfacial models exhibiting broad phase-coexistence. A general discussion as well as the conclusions are given in the last section.

2 Brief review of Toom's NEC model

In the following we describe very briefly Toom's NEC (North-East-Center) model and discuss some of its basic properties. More detailed descriptions may be found in the original papers by Toom [1] and [2,3,5].

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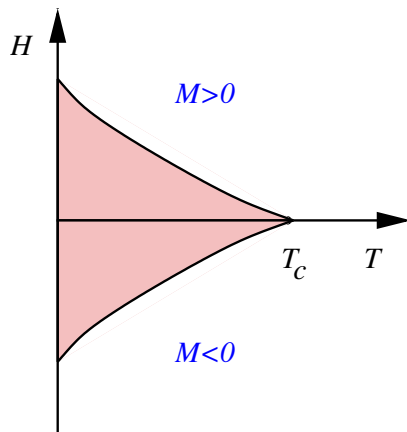


Fig. 1. Phase diagram of Toom's model, showing the phases with positive and negative "magnetization" (M) as well as the broad coexistence region.

The model is defined on a two-dimensional square lattice, with sites occupied by a spin-like variable, $s_i = \pm 1$. The state of the system evolves by simultaneous updating of the spins according to the following rules: i) The value of any given spin is determined by the majority rule applied to a neighborhood that includes the spin itself (C=center) and its neighbors to the north (N) and to the east (E). ii) If as a result of (i) the spin points up (down), it is inverted with probability q (p). These rules are iterated leading eventually to a statistically stationary state. Different types of boundary conditions may be implemented but here we consider periodic boundary conditions only.

Toom's NEC model has two parameters, q and p . Obviously, in the symmetric situation $p = q$ both up and down spins are equally favored, while for $p > q$ ($q > p$) up (down) spins are preferred. A bias field H may be defined, $H = (p - q)/(p + q)$, as the analog of the external magnetic field in the standard Ising model. Likewise, the noise intensity is measured by the temperature-like parameter $T = p + q$. The associated phase diagram in the (H, T) plane is shown in Figure 1. The solid lines are first-order phase boundaries between the one-phase region, with positive or negative magnetization (unshaded area), and the two-phase coexistence region (shaded area). The upper and lower phase boundaries merge at a critical point at $(H = 0, T_c)$. We note that the two-phase coexistence region in the Ising model is the segment of line $T < T_c$ at $H = 0$. By contrast, here phases of positive and negative magnetization lose their stability at $-H(T)$ and $H(T)$, respectively, and the coexistence surface is given by $-H(T) < H < H(T)$ for $T < T_c$. In particular, the lower the temperature, the larger the value of $|H|$ at the first-order phase boundaries, implying that at low intensity of noise the system can sustain phase coexistence even for relatively large values of the bias. This is indeed surprising: In the presence of a relatively large bias, a phase with the opposite magnetization may be truly stable.

Inspection of the dynamical rules leads immediately to the conclusion that, in the deterministic limit $T = 0$, an

initially horizontal interface, separating two semi-infinite planes of up and down spins, will remain immobile. The same will happen if the interface is vertical, or oriented at 45° along the NE-SW diagonal. On the other hand, if the interface is oriented at -45° along the NW-SE diagonal, it will move in a direction perpendicular to it, with constant velocity, and the phase to the right of the interface will advance downwards "eroding" the phase to the left. Thus, any island of, say, up spins in a sea of down spins is effectively eliminated since it can be inscribed in a right triangle with the hypotenuse along the NW-SE diagonal. If noise and bias are switched on, the situation does not change much: the phase to the right of the interface will advance even if it is unfavored by the bias, provided that both T and H are sufficiently small. Upon further increase of the parameters, a point is reached where the unfavored phase is no longer stable, signaling the end of the broad coexistence region [3].

The existence of a broad coexistence region in Toom's model has been rigorously demonstrated [1]. It was also shown that this feature does not depend on the discreteness of the spin variables or of the space-time [5]. The only relevant ingredient seems to be the spatial asymmetry of the nonequilibrium rules. These rules act differently on the interfacial motion depending on the initial orientation of the interface. As a result, islands of spins of the minority phase are unstable even if the phase itself is favored by the bias field.

As a final remark, we mention that the nature of the fluctuations of NEC interfaces separating up- from down-spin regions, was investigated and was found to be related to *Kardar-Parisi-Zhang* (KPZ) [6, 7] nonequilibrium dynamics in the biased case, and to (equilibrium) *Edwards-Wilkinson* (EW) relaxation dynamics [6, 7] in the unbiased one [8].

3 Nonequilibrium depinning of a bound interface

The second example to be discussed arises in the context of nonequilibrium bound interfaces, and has relevance in nonequilibrium wetting, synchronization transitions in extended systems, and general pinning-depinning transitions [9–13]. Most of the material presented in this section is already known, but for the sake of completeness we include it here.

Consider an interface separating two bulk phases, A and B (see Fig. 2). Let a be the chemical potential difference between these two phases, D the surface tension of the AB interface, and $h(\mathbf{x}, t)$ the local height measured from a binding wall or substrate. The interaction between the latter and the interface is usually modeled by a Morse potential

$$V(h) = be^{-h} + e^{-2h}/2, \quad (1)$$

where the repulsive term restricts the interface from fluctuating into the unphysical region $h < 0$, as shown in Figure 3. In the absence of conservation laws, the most

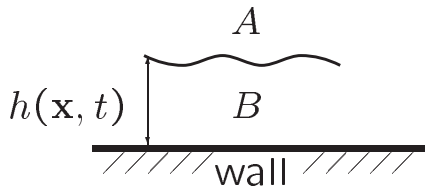


Fig. 2. Interface between the A and B phases, at a distance from the wall.

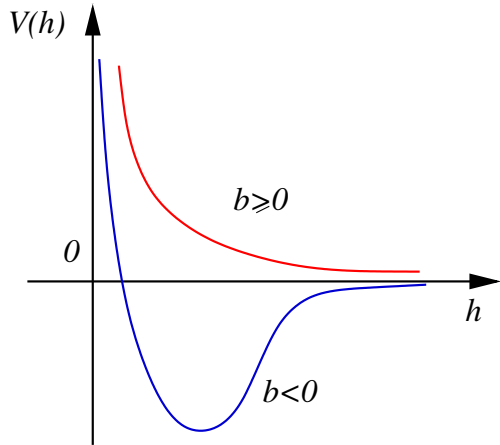


Fig. 3. Binding potential for positive and negative values of b . For $b < 0$ the potential is attractive, exhibiting a local minimum near the wall.

generic nonequilibrium interfacial equation is

$$\partial_t h(\mathbf{x}, t) = D\nabla^2 h + \lambda(\nabla h)^2 + a - \frac{\partial V(h)}{\partial h} + \eta(\mathbf{x}, t), \quad (2)$$

where η is a Gaussian white noise term, that accounts for the thermal fluctuations, and λ is the strength of the KPZ non-linear term, which acts as an external force pushing the tilted interfacial regions against the wall. Now, two different physical situations may occur depending on the sign of b [14, 15].

For $b \geq 0$, the wall is purely repulsive. As a function of a , a continuous, nonequilibrium phase transition from a pinned to a moving interface will occur. Here, we shall not consider this transition. It has been studied extensively and the critical exponents, scaling functions, etc, are well known (see [14, 15] for recent reviews). Assume then that b is negative [16]. In this case, the potential $V(h)$ includes an attractive term (see Fig. 3) which describes the affinity of the substrate for the A phase [9]. Such a term is also required in the context of synchronization transitions [13]. Under these conditions, it can be argued that pinned and depinned phases lose their stability at different points, provided that $\lambda < 0$. For the depinned phase this happens at $a = a^+$ for any value of b (Fig. 4). To see this, notice that as far as their long-time properties are concerned, depinned interfaces may be considered effectively free. Thus, the wall potential may be neglected implying

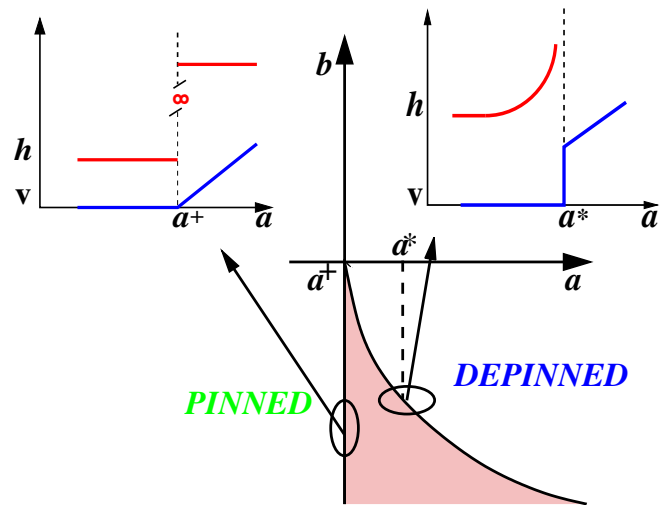


Fig. 4. Phase diagram as a function of a and b . A broad phase coexistence region (shaded area) is observed. Insets: order-parameters (average velocity and average height) at the phase boundaries delimiting the broad coexistence region. In the leftmost (rightmost) panel the height (velocity) changes discontinuously while the velocity (height) changes continuously.

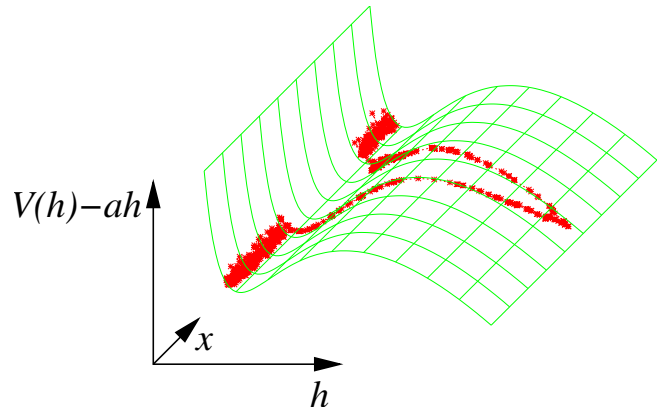


Fig. 5. Typical potential and pinned interface, close to the pinning-to-depinning transition line (rightmost transition in the previous figure). Sites away from the potential-well are pulled by the positive linear term $a > a^+$. In this way, characteristic triangular fluctuations appear and are eliminated, in a region where the depinned phase is also stable.

that the average interfacial velocity $\langle v \rangle$ is independent of b ; this implies in turn that the locus of the depinning transition, where $\langle v \rangle$ changes from positive (depinned) to zero (pinned), is also independent of b . Note that the transition is continuous in terms of the usual order-parameter, $\langle v \rangle$, but discontinuous in terms of the average interface position, $\langle h \rangle$, that jumps from infinity (depinned) to a finite value close to the potential minimum (pinned).

On the other hand, the pinned phase loses its stability when the density of pinned sites, i.e. sites within the attractive potential well, vanishes. This happens at a value

$a^*(b) > a^+$ which depends on b . It has been shown [13,14] that this depinning transition is related to *directed percolation* (DP) [17]. In Figure 5 we plot a typical pinned interface within the coexistence region. Most of the sites lie within the potential well. Those patches which, owing to fluctuations, overcome the potential barrier are locally depinned and grow quickly to form triangular structures, with a well defined slope, anchored at the wall (Fig. 5). Driven by the negative nonlinear term, $\lambda(\nabla h)^2$, these triangles shrink at a constant velocity revealing the mechanism for the elimination of islands of the depinned phase. Therefore, even if the depinned phase is stable, initially pinned interfaces will remain so, since a mechanism for the elimination of islands of the depinned phase does exist. By contrast, for systems with $\lambda > 0$, broad phase coexistence does not occur because triangular fluctuations are pulled away from the wall [14,18] and thus there is no mechanism to eliminate the minority phase islands.

As the stability threshold $a^*(b)$ is approached, the size of the depinned regions increases until they extend over the whole system. Then the triangular fluctuations cannot be eliminated and the interface is depinned. Consequently, for $b < 0$ pinned and depinned phases lose their stability at different values of the control parameter a . Between these values, there is a broad region of phase-coexistence where initially pinned interfaces remain pinned, while moving interfaces keep on moving. This remains the case in the infinitely-large system-size limit [19].

Let us stress that while the transition at a^* is controlled by the effect of fluctuations that make pinned sites jump over the potential well, the one at a^+ is controlled by the average velocity of the free interface. Therefore two different mechanisms are at play. *Essential for the broad phase-coexistence region is the asymmetric role played by the potential on pinned and depinned interfaces: the potential has no effect on depinned interfaces, while it stabilizes pinned interfaces in a region where depinned interfaces are also stable. The potential acts as an asymmetric force that depends on the state of the interface.*

The dynamical asymmetry of the potential, however strong, cannot by itself produce generic bistability. A nonequilibrium ingredient is also needed, as can be seen by letting $\lambda = 0$ in equation (2). This results in the equilibrium EW equation in the presence of an attractive potential, which describes the pinning-depinning transition of an equilibrium interface where no broad phase-coexistence exists. As in Toom's model, it is the combination of the nonequilibrium nature and the asymmetry of the dynamics that provides the mechanism for the elimination of minority phase islands, required for the existence of a broad region of phase coexistence.

Before ending this section, let us remark that the phenomenology described above is not unique to the solutions of the continuum equation (2); it was first noticed in discrete models of nonequilibrium wetting [18]. In addition, broad phase-coexistence is still observed when long-ranged potentials or slightly different nonequilibrium dynamics are considered.

4 Pinning-depinning in disordered media

The third example to be discussed concerns the transitions between static and moving interfaces in the presence of quenched disorder. Relevant applications are ubiquitous and include solid-on-solid friction experiments, charge density waves [20], wetting of rough surfaces [21], vortex lines in type II superconductors [22], and earthquakes [23], to mention but a few.

A familiar case is provided by elastic, extended objects sliding against rough surfaces. Segments of the sliding object may be pinned by the inhomogeneities of the medium, only to be set into motion by the elastic forces from neighboring regions that have overcome the resistance of the medium and move freely. The final, average velocity will depend on the strength of the applied external force. A simplified model that purports to capture this phenomenology is the ‘‘Quenched Edwards Wilkinson’’ (QEW) equation (see [6,7])

$$\partial_t h(\mathbf{x}, t) = D\nabla^2 h(\mathbf{x}, t) + F + \eta(\mathbf{x}, h(\mathbf{x}, t)). \quad (3)$$

This equation describes an elastic interface, given by the height profile $h(\mathbf{x}, t)$, with surface tension D , under the influence of a constant external driving term F , and *quenched noise* $\eta(\mathbf{x}, h(\mathbf{x}, t))$. It exhibits a continuous *pinning-depinning* transition at a critical force F_c from a pinned phase to a moving one. The universal properties of this transition were studied using renormalization group methods [24], and the critical exponents were measured both computationally and experimentally [25,26].

The simplest way to simulate the dynamics of the universality class described by equation (3) is by using the Leschhorn cellular automaton [27]. A quenched random pinning force, $f(x, h(x))$, sampled homogeneously from the interval $[0, 1]$ is assigned to each coordinate $(x, h(x))$, in one-dimension, and an interface profile $h(x, t = 0) = 1$ is taken as the initial condition. The dynamics then proceeds as follows: at every time step, and at every site x , a local force is given by the sum of i) the discretized Laplacian at that site, $(h(x+1) + h(x-1) - 2h(x))$, and ii) the constant force F . If the local force exceeds the pinning value, $f(x)$, then $h(x)$ is increased by one unit. The process is repeated for all sites and iterated in time. Below a given F^* the system is pinned with probability one (in the infinite system-size limit), while for larger values of F the interface advances with constant velocity. This phase transition is in the QEW universality class, as described by equation (3).

It is well known that the force required to start an object moving is greater than that necessary to keep it going. This is not captured by the QEW nor by the Leschhorn automaton due to a *no-passing* rule: an interface cannot overtake another one that is initially ahead of it. This is due to the fact that at every point of contact the pinning force is the same for both interfaces, whereas the elastic, restoring force on the advanced interface is greater than (or equal) the restoring force on the interface lagging behind. As a consequence of this rule, coexistence of moving and stationary interfaces is impossible since for a given external force the interface attains a unique velocity.

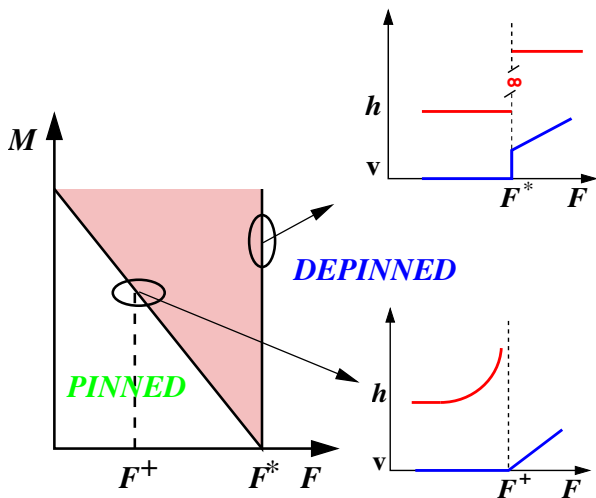


Fig. 6. Phase diagram as a function of F and M . A broad phase-coexistence region (shaded) exists. The order parameters (both average velocity and average height) are plotted for both transitions delimiting the broad coexistence region. In the lowermost (uppermost) panel the height (velocity) changes discontinuously while the velocity (height) changes continuously. The threshold in the F^+ branch, beyond which the transition becomes discontinuous is not shown.

In an attempt to describing more realistic situations, *dynamic stress transfer mechanisms* were introduced recently in the Leschhorn automaton [28]. The local force at every site is increased by an extra contribution $MS(x)$, where M is a control parameter, and $S(x)$, a *stress-overshoot*, is equal to 1 if either the site at x or at any of its nearest neighbors moved in the preceding time-step, and 0 otherwise. In this way, locally moving interfaces are more likely to keep on moving, while the stress-overshoots do not play any role on locally pinned regions, where $S(x) = 0$. This will produce an effect similar to that of inertia [28]. While the original Leschhorn model has a transition at F^* , the model endowed with stress-overshoots can easily be seen to undergo a pinning transition at $F^+ = F^* - M$ (see Fig. 6). To see this, note that at all moving sites the local force is increased by M units, therefore the effective external force is $F + M$ (see [28] for more details). On the other hand the depinning transition is not affected by the stress-overshoots, and therefore remains at F^* . This leads to a broad region of phase-coexistence delimited by $[F^+, F^*]$. The transition at F^+ was reported to be continuous (in terms of velocities) and to belong to the QEW class for small M , and discontinuous above some value of M . The origin of this broad region of phase-coexistence can be traced to the fact that the stress-overshoots only affect the moving phase. Therefore, regions expected to belong to the pinned phase in the original model may become depinned due to the extra force generated by inertia if they are initially moving, but remain pinned if the interface is initially pinned. In a nutshell, *broad coexistence results from the asymmetric role*

played by the stress-overshoots in pinned and depinned interfaces. It should be stressed that the hysteresis in this type of models is a *phony* one, in the sense that it is destroyed by the inclusion of thermal fluctuations [28].

As in the previous section, asymmetry is only a necessary condition, not a sufficient one; here, a robust mechanism for the elimination of islands of the minority phase is also required. In this case, it is essential that the inertial term depends on the state of motion of the neighboring sites as well as on the site itself. The mechanism, however, seems to be sensitive to the details of the model [28, 29].

Other universality classes in the interfaces-in-random-media realm are the quenched KPZ equation [6, 7, 30], the EW equation with columnar noise [6, 31], or the Mullins-Herring equation [32]. All of them are susceptible of exhibiting broad phase coexistence by an adequate inclusion of inertial effects.

5 Discussion and conclusions

Toom's NEC probabilistic automaton, introduced some 20 years ago, exhibits generic bistability. Its relation to recently proposed, nonequilibrium models should be noticed inasmuch as these models also exhibit broad phase-coexistence. In the latter, realistic driving forces, a and F , favor interfacial motion, and are opposed by pinning forces, the attractive wall and the random medium, respectively. The latter depend crucially on the interfacial state.

The key feature of Toom's NEC model is the spatial *asymmetry of its dynamical rules*, of a type only possible in nonequilibrium systems, that results in an efficient *mechanism for the elimination of islands of the minority phase*. Indeed, islands of minority spins shrink after they have been created by fluctuations at a velocity roughly independent of their radius. This is to be compared with the Ising model in zero magnetic field, where a droplet of radius r shrinks with a velocity proportional to $1/r$.

These two ingredients are also present, in more realistic terms, in the interfacial models discussed in this paper, namely, a nonequilibrium interface bound by an attractive wall (Eq. (2)), and an interface subject to stress-overshoots advancing in a random medium (Eq. (3)). In particular,

i) There is a clear asymmetry in both cases: the potential well acts only on locally pinned regions, and inertia (stress-overshoots) does so only on locally moving ones. Therefore, in one of the coexisting phases the additional 'force' plays no role. This dynamical asymmetry is the first essential ingredient for broad phase-coexistence. Both the attractive potential and the stress-overshoots have a clear physical origin, and play the same role as the somewhat artificial spatial asymmetry of Toom's model.

ii) A robust mechanism for the elimination of islands of the stable phase may be found in both examples. In the first case, droplets of the depinned phase, that could make the interface detach from the wall, acquire a triangular shape and are ultimately suppressed by the combination of the nonlinear force term $\lambda < 0$ and the wall. The islands

of depinned sites cannot act as nucleation bubbles and are eliminated in a time proportional to their size. Similarly, stress-overshoots foster the depinning of segments of the interface, when $F^+ < F < F^*$, preventing nucleation of the pinned interface. As explained in the paper, modifications of these nonequilibrium models that fail to provide such a mechanism, do not exhibit broad phase coexistence.

Note that while in the bound-interface model, it is the *pinned* phase that is further stabilized by the asymmetric dynamics, in the stress-overshoots model it is the *depinned* phase that is further stabilized.

Another interesting issue concerns the nature and the universality class of the continuous transitions. Interestingly enough, for the model with stress-overshoots the continuous (in terms of $\langle v \rangle$) pinning transition, in the QEW universality class as in the original inertia-free Leschhorn automaton, persists for small M , whereas for larger M it becomes discontinuous. On the other hand, despite previous claims, for the model of bound interfaces the depinning transition remains continuous (in terms of $\langle h \rangle$), but the universality class changes from the so-called *multiplicative noise* class [14] for $b > 0$, to directed percolation for $b < 0$ (attractive wall).

In conclusion, *the additional term required to generate broad phase-coexistence can be a relevant or an irrelevant perturbation to the continuous phase transition of the original model.* Moreover, the order of the transition may or may not be affected by this term.

Finally, let us stress again, that in both interfacial models, one of the transitions delimiting the broad coexistence region can be continuous, at odds, with the intuition developed in equilibrium situations. Nevertheless, at least one of the boundaries has to be discontinuous, as can be checked easily using continuity arguments (see Figs. 4 and 6).

Summing up, two interfacial models exhibiting generic bistability have been discussed. Both are nonequilibrium models with an essential asymmetry in the dynamics. The asymmetry is such that it eliminates islands of the minority phase efficiently. The stability of one of the phases is enhanced, resulting in generic bistability over a broad region of parameter space. In both models this is achieved by including realistic ingredients, namely an attractive wall and inertia. This puts generic phase coexistence, with its many conceptual and applied consequences, under a more solid and motivated physical basis.

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